Homework 9
STA4322/STA5328
Introduction to Statistics Theory
Spring 2020, MWF 3:00pm
Due date: Friday, April 17th, 2020 at 5:00pm

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*. Problem 3c is worth 1pt of extra credit.

1. Suppose that $Y_1, \ldots, Y_n$ denote a random sample from the probability density function given by

$$f_Y(y \mid \theta, \lambda) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-\frac{(y-\lambda)}{\theta}} & y > \lambda \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the likelihood ratio test statistic for testing $H_0 : \theta = \theta_0$ versus $H_A : \theta > \theta_0$ with λ unknown.

   (b) Assuming n is sufficiently large and the "regularity conditions" are satisfied, use Wilk's theorem to find the rejection region of an asymptotic level α test for the hypotheses in (a).

2. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed exponential random variables with mean β_X and $Y_1, \ldots, Y_n$ are independent and identically distributed exponential random variables with mean β_Y. Recall that this the density for an exponentially distributed random variable U with mean β,

$$f_U(u) = \frac{1}{\beta} e^{-\frac{u}{\beta}} \mathbf{1}(u > 0).$$

   We wish to test the hypotheses

$$H_0 : \beta_X = \beta_Y, \quad \text{vs} \quad H_A : \beta_X \neq \beta_Y.$$

   Let $\Theta = (\beta_X, \beta_Y)$. Let $L(x, y \mid \Theta)$ be the joint likelihood of $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ evaluated at Θ.

   (a) What is $\max_{\Theta \in \Omega_0} L(x, y \mid \Theta)$?

   (b) What is $\max_{\Theta \in \Omega_0 \cup \Omega_A} L(x, y \mid \Theta)$?

   (c) Show that the likelihood ratio test rejects H_0 if

$$\frac{\left(\sum_{i=1}^n X_i\right)\left(\sum_{i=1}^n Y_i\right)}{\left[\sum_{i=1}^n (X_i + Y_i)\right]^2} < k.$$

   No need to find k, you must only show that the rejection region has the form above.

3. A random sample $W_1, \ldots, W_n$ is drawn from a distribution with density

$$f_W(w) = \frac{\theta \nu^\theta}{w^{\theta+1}} 1(w \geq \nu)$$

   where $\theta > 0$ and $\nu > 0$ are unknown parameters.

(a) Find the MLE of θ and ν.

(b) Give the likelihood ratio test statistic λ for the hypotheses

$$H_0 : \theta = 1, \nu > 0 \quad \text{vs} \quad H_A : \theta \neq 1, \nu > 0.$$

(c) (Extra credit) Show that the LRT for

$$H_0 : \theta = 1, \nu > 0 \quad \text{vs} \quad H_A : \theta \neq 1, \nu > 0$$

has rejection region of the form

$$\{W : T(W) \leq c_1 \cup T(W) \geq c_2\}$$

where $0 < c_1 < c_2$ are some constants (no need to find them) and

$$T(W) = \log \left[\frac{\prod_{i=1}^n W_i}{(W_{(1)})^n} \right],$$

where $W_{(1)} = \min(W_1, \ldots, W_n)$.