Homework 8
STA4322/STA5328
Introduction to Statistics Theory
Spring 2020, MWF 3:00pm
Due date: Monday, April 13th, 2020 at 5:00pm

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*.

1. A random sample of 37 first graders who participated in sports had manual dexterity scores with mean 32.19 and standard deviation 4.34. An independent sample of 37 first graders who did not participate in sports has manual dexterity scores with mean 31.68 and standard deviation 4.56. Assume that both distributions are normal with equal variance and possibly distinct means.

   (a) Test whether there is sufficient evidence to conclude that first graders who participated in sports have a higher dexterity score at the 0.05 level. It may be useful to know that $t_{72,0.95} \approx 1.67$.

   (b) For the rejection region used in (a), calculate the type II error probability β when $\mu_1 - \mu_2 = 3$ (i.e., the dexterity score mean of those who participated in sports is 3 points higher). You may leave your answer in terms of a probability statement involving a particular t-distribution.

2. A publisher of a magazine has found through past experience that 60% of subscribers renew their subscriptions. In a random sample of 200 subscribers, 108 indicated that they planned to renew their subscriptions. What is the p-value associated with the test that the current rate of renewals differs from the rate previously experienced? You may leave your answer in terms of summations.

3. Suppose that $X_1, \ldots, X_n$ are a random sample from a normal distribution with known mean $\mu = 0$ and unknown variance.

   (a) Find the most powerful α-level test of

   $$H_0 : \sigma^2 = \sigma_0^2 \quad \text{vs} \quad H_A : \sigma^2 = \sigma_A^2,$$

   where $\sigma_A^2 > \sigma_0^2$. It may be useful to recall that $\frac{1}{\sigma^2}\sum_{i=1}^{n} X_i^2 \sim \chi_n^2$.

   (b) Is this test uniformly most powerful for $H_A : \sigma^2 > \sigma_0^2$? How do you know?

4. Suppose we want to collect a random sample from a normal distribution with $\sigma^2 = 25$ and unknown mean μ to test $H_0 : \mu = 10$ versus $H_A : \mu = 5$. What is the sample size n needed so that the most powerful test for H_0 versus H_A will have $\alpha = \beta = 0.025$ (i.e., will have both type I and type II error probability 0.025).

5. Suppose Y is a random variable having a distribution with density

$$f_Y(y) = \begin{cases} \theta y^{\theta-1} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$.

   (a) Sketch the power function of the test with rejection region $Y > 0.5$. Note that the power function is a function of θ_A where $\mathrm{Power}(\theta_A) = 1 - \beta(\theta_A)$ with $\beta(\theta_A)$ being the type II error probability at θ_A.

   (b) Based on a random sample of size $n = 1$, find a uniformly most powerful test of size α for testing

$$H_0 : \theta = 1 \quad \text{vs} \quad H_A : \theta > 1.$$

6. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $X_i \sim \mathrm{Poisson}(\lambda)$ for $i = 1, \ldots, n$. It could be useful to recall that $\sum_{i=1}^n X_i \sim \mathrm{Poisson}(n\lambda)$.

   (a) Find the form the rejection region for a most powerful test of

$$H_0 : \lambda = \lambda_0 \quad \text{vs} \quad H_A : \lambda = \lambda_A$$

   for $\lambda_A > \lambda_0$.

   (b) Is the test derived in (a) uniformly most powerful for testing $H_0 : \lambda = \lambda_0$ versus $H_A : \lambda > \lambda_0$? How do you know?

   (c) Find the form the rejection region for a most powerful test of

$$H_0 : \lambda = \lambda_0 \quad \text{vs} \quad H_A : \lambda = \lambda_A$$

   for $\lambda_A < \lambda_0$.

7. Suppose $U_1, \ldots, U_n$ are independent and identically distributed random variables where each U_i has the exponential distribution with mean θ.

   (a) Derive the most powerful test for $H_0 : \theta = \theta_0$ again $H_A : \theta = \theta_A$ where $\theta_A < \theta_0$. You may find it useful to recall that the sum of iid exponentially distributed random variables follows a particular Gamma distribution.

   (b) Is the test derived in part (a) uniformly most powerful for testing $H_0 : \theta = \theta_0$ against $H_A : \theta < \theta_0$?