Homework 6
STA4322/STA5328
Introduction to Statistics Theory
Spring 2020, MWF 3:00pm
Due date: Monday, March 16th, 2020

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*. Question 4c is worth 2 pts of extra credit.

1. Find the maximum likelihood estimator of θ based on the random sample of size n from a Uniform$(0, 2\theta)$ distribution.

2. Suppose that $X_1, X_2, \dots, X_n$ are independent and identically distributed random variables with density

$$f_X(x \mid \theta) = \begin{cases} e^{-(x-\theta)} & x > \theta \\ 0 & \text{otherwise} \end{cases}$$

   where $\theta > 0$ is an unknown constant.

   (a) Find an estimator $\hat{\theta}_1$ for θ using the method of moments.

   (b) Find the maximum likelihood estimator (MLE) of θ, $\hat{\theta}_2$.

   (c) Modify both $\hat{\theta}_1$ and $\hat{\theta}_2$ so that they are unbiased for θ.

   (d) Using the modified versions from part (c), compute the efficiency of the modified version of $\hat{\theta}_2$ relative to the modified version of $\hat{\theta}_1$.

3. Suppose that m integers are randomly drawn *with replacement* from the integers $1, 2, \dots, M$. That is, each integer has probability $1/M$ of taking on any values $1, 2, \dots, M$, and the sampled values are independent.

   (a) Find the method of moments estimator of $\hat{M}$ of M.

   (b) Compute $\mathrm{E}(\hat{M})$ and $\mathrm{Var}(\hat{M})$.

4. The geometric distribution has probability mass function

$$P(Y = y) = p(1-p)^y.$$

   Suppose we have observed a random sample of size n from the geometric distribution with unknown parameter p.

   (a) Find the method of moments estimator of p.

   (b) Find the MLE of p.

   (c) (Extra credit) Using a sufficient statistic, find an unbiased estimator of p. Note that you will have to use the Rao-Blackwell theorem directly (i.e., conditioning on the sufficient statistic). For example, see the steps of question 9.65 of the textbook.

5. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with density

$$f_X(x \mid \theta) = \begin{cases} \frac{3x^2}{\theta^3} & 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $X_{(n)} = \max(X_1, \ldots, X_n)$ is a sufficient statistic.

(b) Find the MLE of θ.

(c) Find a function of the MLE which is a pivotal quantity.

(d) Use the pivotal quantity to construct a $100(1-\alpha)$% confidence interval for θ.