**Homework 4**
STA4322/STA5328
Introduction to Statistics Theory
Spring 2020, MWF 3:00pm
Due date: Friday, February 21st, 2020

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*.

1. Let $X_1, \dots, X_n$ be independent and identically distributed random variables, each with the density

$$f_X(x \mid \alpha, \beta) = \begin{cases} \alpha\beta^{\alpha}x^{-(\alpha+1)} & : x \geq \beta \\ 0 & : \text{otherwise} \end{cases}$$

   Suppose $\beta$ is known. Find a sufficient statistic for $\alpha$.

2. Let $U_1, \dots, U_n$ be a random sample from the $\text{Uniform}(\theta_1, \theta_2)$ distribution. Show that $U_{(n)} = \max(U_1, \dots, U_n)$ and $U_{(1)} = \min(U_1, \dots, U_n)$ are jointly sufficient for $\theta_1$ and $\theta_2$.

3. Let $Y_1, \dots, Y_n$ be independent and identically distributed random variables with density:

$$f_Y(y \mid \theta) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{-y^2/\theta} & : y > 0 \\ 0 & : \text{otherwise} \end{cases}$$

   (a) What is the likelihood $L(y_1, \dots, y_n \mid \theta)$?
   (b) Find a sufficient statistic for $\theta$ by factorizing the likelihood into the product of two functions: one depending only on the sufficient statistic and $\theta$; the other depending only on the data.
   (c) Assuming the sufficient statistic you have found is minimal, find a minimum variance unbiased estimator of $\theta$.

4. Let $W_1, \dots, W_n$ be independent and identically distributed random variables $\text{Binomial}(m, p)$ random variables.
   (a) Find a sufficient statistic for $p$ using the entire data $(W_1, \dots, W_n)$.
   (b) Assuming your sufficient statistic is minimal, find the minimum variance unbiased estimator of $p(1-p)$.

5. Let $Y_1, \dots, Y_n$ be a random sample from a population with density

$$f_Y(y \mid \theta) = \begin{cases} \frac{3y^2}{\theta^3} & : 0 < y < \theta \\ 0 & : \text{otherwise} \end{cases}$$

   (a) Show that $Y_{(n)} = \max(Y_1, \dots, Y_n)$ is a sufficient statistic for $\theta$.
   (b) Given that $Y_{(n)}$ is a minimal sufficient statistic, find the MVUE of $\theta$.

6. If $X_1, \ldots, X_n$ be independent and identically distributed random variables following a Beta$(\alpha, \beta)$ distribution, i.e., the density of each $X_i$ is

$$f_X(x \mid \alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & : 0 < x < 1 \\ 0 & : \text{otherwise} \end{cases}$$

(a) Suppose $\beta$ is known. Find a sufficient statistic for $\alpha$.

(b) Again supposing $\beta$ is known, find the method of moments estimator of $\alpha$.