Homework 3
STA4322/STA5328
Introduction to Statistics Theory
Spring 2020, MWF 3:00pm
Due date: Friday, February 7th, 2020

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*.

1. Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Poisson distribution with mean λ. Consider $\hat{\lambda}_1 = (Y_1 + Y_2)/2$ and $\hat{\lambda}_2 = \bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$. Derive the relative efficiency of $\hat{\lambda}_1$ relative to $\hat{\lambda}_2$.

2. Let $W_1, W_2, \ldots, W_n$ denote a random sample from the probability density function
$$f_W(w) = \begin{cases} \theta w^{\theta-1} & : 0 < w < 1 \\ 0 & : \text{otherwise} \end{cases}$$
where $\theta > 0$. Show that $\bar{W} = \frac{1}{n}\sum_{i=1}^n W_i$ is a consistent estimator of $\theta/(\theta+1)$.

3. Suppose that $Y_1, Y_2, \ldots, Y_n$ is a random sample where $Y_i \sim \mathrm{Poisson}(\lambda)$ for all $i = 1, \ldots, n$. Assume that $n = 2k$ for some integer k (i.e., assume that n is even). Let
$$\hat{\lambda} = \frac{1}{2k}\sum_{i=1}^{k}(Y_{2i} - Y_{2i-1})^2.$$
   (a) Show that $\hat{\lambda}$ is an unbiased estimator of λ.
   (b) Show that $\hat{\lambda}$ is a consistent estimator of λ.

4. Let $Z_1, Z_2, \ldots, Z_n$ be independent standard normal random variables. In the previous homework, we used the that $Z_i^2 \sim \chi_1^2$ for $i = 1, \ldots, n$ and because the Z_i's are independent $Z_1^2 + Z_2^2 + \cdots + Z_n^2 \sim \chi_n^2$. Now, let us define
$$W_n = \frac{1}{n}\sum_{i=1}^{n} Z_i^2.$$
Does W_n converge in probability to some constant c? If so, what is the value of c?

5. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with density
$$f_Y(y) = \begin{cases} 3y^2 & : 0 \le y \le 1 \\ 0 & : \text{otherwise} \end{cases}$$
Show that $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$ converges in probability to some constant and find the constant.

6. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from an exponential distribution with mean θ.

   (a) Show that $\hat{\theta}_1 = nX_{(1)} = n \min(X_1, \ldots, X_n)$ is an unbiased estimator of θ.

   (b) Show that $\mathrm{MSE}(\hat{\theta}_1) = \theta^2$.

   (c) Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2 = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.