**Homework 1**
STA4322/STA5328
Introduction to Statistics Theory
Spring 2020, MWF 3:00pm
Due date: Wednesday, January 22nd, 2020

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*.

1. Let $X_1, X_2, \ldots, X_n$ be independent random variables following the uniform distribution on the interval $[0, \theta]$ for some $\theta > 0$..
   (a) Find the probability distribution function of $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$.
   (b) Find the probability density function of $X_{(n)}$.
   (c) Compute the mean and variance of $X_{(n)}$.

2. As in Problem 1, let $X_1, X_2, \ldots, X_n$ be independent random variables following the uniform distribution on the interval $[0, 1]$. Find the density of $X_{(k)}$, the $k$th order statistic, where $k$ is an integer between $1$ and $n$.

   **Hint:** You may use the fact that

$$\frac{d}{dp}\sum_{j=k}^{n}\binom{n}{j}p^j(1-p)^{n-j} = n\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}$$

   without proof.

3. (WMS 8.12) Let $Y_1, \ldots, Y_n$ be a random sample where for $i = 1, \ldots, n$, we know $Y_i \sim \mathrm{Uniform}(\theta, \theta + 1)$ where $\theta > 0$ is an unknown parameter.
   (a) Show that $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is a *biased* estimator of $\theta$.
   (b) Propose an estimator of $\theta$ which is unbiased. Verify that this estimator is unbiased.
   (c) Compute the MSE of $\bar{Y}$ with respect to $\theta$, i.e., compute $\mathrm{MSE}_\theta(\bar{Y})$

4. Suppose that the random variables $X_1$ and $X_2$ are independent and that both have the normal distribution with mean $\mu$ and variance $\sigma^2$. Prove that $\bar{X} = \frac{1}{2}\sum_{i=1}^{2} X_i$ and $s^2 = \frac{1}{2-1}\sum_{i=1}^{2}(X_i - \bar{X})^2$ are independent.

**Hint:** Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Show that $\bar{X}$ and $s^2$, respectively, can be written in terms of $Y_1$ and $Y_2$ alone. Then, verify that $Y_1$ and $Y_2$ are independent using that if two normally distributed random variables have zero covariance, they are independent.

5. Suppose that a random sample $X_1, X_2, \ldots, X_n$ is to be taken from the uniform distribution on the interval $[0, \phi]$ with $\phi > 0$ unknown. What $n$ is required so that

$$P\left(|\max(X_1, X_2, \ldots, X_n) - \phi| < 0.1\phi\right) \geq 0.95$$

for all possible $\phi$?