Homework 9
STA 4321/5325, Fall 2019, MWF 8:30am
Professor: Aaron J. Molstad
Due date: Monday, November 25th, 2019

All work must be shown for complete credit. W.M.S. denotes the course textbook, *Mathematical Statistics with Applications*. Problem #8 is worth 2pts of extra credit.

1. Let the joint density function of random pair (U, V) be

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{4}(u+2v) & 0 < v < 1,\ 0 < u < 2 \\ 0 & \text{otherwise} \end{cases}$$

   In Homework 8, we showed

$$f_U(u) = \begin{cases} \frac{(u+1)}{4} & 0 < u < 2 \\ 0 & \text{otherwise} \end{cases}$$

   What is the density function of the random variable $Z = 9/(U+1)^2$?

2. Let Z be a standard normal random variable and let $Y_1 = Z$ and $Y_2 = Z^2$.

   (a) What are $\mathrm{E}(Y_1)$ and $\mathrm{E}(Y_2)$?

   (b) What is $\mathrm{E}(Y_1 Y_2)$?
   (Hint: $\mathrm{E}(Y_1 Y_2) = \mathrm{E}(Z^3)$ and the MGF of the standard normal is $M_Z(t) = e^{\frac{t^2}{2}}$.)

   (c) What is $\mathrm{Cov}(Y_1, Y_2)$?

   (d) Are Y_1 and Y_2 independent? If not, provide some information which contradicts their independence.

3. A worker leaves for work between 9:00am and 9:30am and takes between 45 and 55 minutes to arrive. Let the random variable Y denote this worker's time of departure, and the random variable X the travel time. Assuming that Y and X are independent and uniformly distributed, find the probability that the worker arrives at work before 10:00am.

4. Suppose the distribution of X, conditional on $U = u$, is $\mathrm{N}(u, u^2)$ where the marginal distribution of U is uniform $(0, \theta)$.

   (a) Find $\mathrm{E}(X)$, $\mathrm{Var}(X)$, and $\mathrm{Cov}(X, U)$.

   (b) Prove that X/U and U are independent.

5. (W.M.S 6.29) The speed of a molecule in a uniform gas at equilibrium is a random variable V whose density function is given by

$$f_V(v) = av^2 e^{-bv^2}, \quad v > 0$$

where $b = m/2kT$ with $k, T,$ and m denoting Boltzmann's constant, the absolute temperature, and the mass of the molecule, respectively, and a is some normalizing constant (so that the density integrates to one).

   (a) Derive the distribution of $W = mV^2/2$, the *kinetic energy* of the molecule. (Hint: what a makes W a Gamma random variable?)

   (b) Find $\mathrm{E}(W)$.

6. Let X have a distribution function given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^2} & x \geq 0 \end{cases}$$

Find a transformation $g(U)$ such that if U has a uniform distribution on the interval $(0, 1)$, $g(U)$ has the same distribution as X.

7. Let X_1 and X_2 be two random variables with joint density

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} e^{-x_1} & 0 \leq x_2 \leq x_1 < \infty \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute $P(X_1 < 2, X_2 > 1)$.

   (b) Compute $P(X_1 \geq 2X_2)$.

   (c) Compute $P(X_1 - X_2 \geq 1)$

8. (Optional) (W.M.S, 6.52) Let V_1 and V_2 be independent Poisson random variables with means λ_1 and λ_2, respectively.

   (a) What is the probability distribution function of $V_1 + V_2$?

   (b) What is the conditional probability function of Y_1 given that $Y_1 + Y_2 = m$?