Homework 8
STA 4321/5325, Fall 2019, MWF 8:30am
Professor: Aaron J. Molstad
Due date: Monday, November 18th, 2019

All work must be shown for complete credit.

1. Suppose that a pair of random variables (X, Y) is uniformly distributed on the vertices of the square $[-1, 1] \times [-1, 1]$: i.e., the joint mass function has nonzero probability on $(-1, -1)$, $(1, -1)$, $(-1, 1)$, and $(1, 1)$ with each of these four points occurring with probability $p_{X,Y}(x, y) = \frac{1}{4}$.
   (a) Compute $P(X^2 + Y^2 < 1)$
   (b) Compute $P(2X - Y > 0)$
   (c) Compute $P(|X - Y| < 2)$

2. Suppose that the random variables Y_1, Y_2 have joint probability density function given by

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6y_1^2 y_2 & 0 \leq y_1 \leq y_2, y_1 + y_2 \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that the marginal density of Y_1 is a beta distribution with parameters $\alpha = 3$ and $\beta = 2$.
   (b) Derive the marginal density of Y_2.
   (c) Derive the conditional density of Y_2 given $Y_1 = y_1$, $f_{Y_2|Y_1=y_1}$.
   (d) Find $P(Y_2 < 1.1 \mid Y_1 = 0.6)$

3. Let the random pair (X_1, X_2) have density

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 4x_1 x_2 & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute $\mathrm{E}(X_1)$
   (b) Compute $\mathrm{Var}(X_1)$
   (c) Compute $\mathrm{E}(X_1 - X_2)$

4. Let the joint density function of random pair (U, V) be

$$f_{U,V}(u, v) = \begin{cases} C(u + 2v) & 0 < v < 1,\ 0 < u < 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the value of C such that $f_{U,V}$ is a valid density.
   (b) What is f_U, the marginal density of U?

(c) Find the joint probability distribution function $F_{U,V}$ of (U, V).

(d) (Moved to HW9) What is the density function of the random variable $Z = 9/(U+1)^2$?

5. Let the joint density of (V_1, V_2) be

$$f_{V_1,V_2}(v_1, v_2) = \begin{cases} \frac{1}{8} v_1 e^{-(v_1+v_2)/2} & v_1 > 0, \quad v_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Prove that V_1 and V_2 are independent.

(b) If V_1 and V_2 both represent the lifetime of an object, we may measure their efficiency in terms of the ratio V_2/V_1. Find $\mathrm{E}(V_2/V_1)$.