Homework 7
STA 4321/5325, Fall 2019, MWF 8:30am
Professor: Aaron J. Molstad
Due date: Wednesday, October 30th, 2019

All work must be shown for complete credit. Problem #4 is worth 1pt of extra credit.

1. Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma > 0$ (that is, X is normally distributed with mean μ and variance σ^2). Given some realization of X, a mathematician constructs a rectangle with length $L = |X|$ and width $W = 4|X|$. What is the expected value of the area of the rectangle?

2. Let V be a random variable following the beta distribution with parameters α, β. Specifically, the density of V is
$$f_V(v) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1} & : 0 \leq v \leq 1 \\ 0 & : \text{otherwise} \end{cases}$$
Find $\mathrm{E}(V^k)$ for arbitrary integer k without using moment generating functions. You answer may be left in terms of quantities involving the Γ function.

3. Derive the moment generating function of Y, a negative binomial random variable with $r = 10$ and success probability p (i.e., Y is the number of failures before the $r = 10$th success in a sequence of independent Bernoulli trials). You may use the following facts without proof:
   (a) If $X_1, X_2, \dots, X_{10}$ are independent geometric random variables each with success probability p, then we can write $Y = \sum_{i=1}^{10} X_i$.
   (b) If U and V are independent random variables, then for any function $g : \mathbb{R} \to \mathbb{R}$, $\mathrm{E}[g(U)g(V)] = \mathrm{E}[g(U)]\mathrm{E}[g(V)]$ (assuming all relevant expected values exist).
   (c) (Lecture 12) For all constants t and r such that $|r\exp(t)| < 1$
   $$\sum_{k=0}^{\infty} [r\exp(t)]^k = \frac{1}{1 - r\exp(t)}.$$
   Note that when r is positive, $t < -\log(r) \implies |r\exp(t)| < 1$.

4. (Optional) It is common for engineers to work with the "error function"
$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\left(-x^2\right) dx$$
instead of the standard normal probability distribution function Φ, which we defined as:
$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx.$$

Show that the following relationship between Φ and the function erf holds for all z:

$$\Phi(z) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right).$$

5. Let W be a random variable whose probability density function is

$$f_W(w) = \begin{cases} \frac{2w}{\lambda^2}\exp\left[-\left(\frac{w}{\lambda}\right)^2\right] & : w \geq 0 \\ 0 & : \text{otherwise} \end{cases}$$

for parameter $\lambda > 0$. Note that if you cannot solve (a), you should still attempt (b) with the given MGF.

(a) Show that the moment generating function of W, $M_W(t)$ satisfies

$$M_W(t) = \sum_{n=0}^{\infty} \frac{(t\lambda)^n}{n!}\Gamma\left(1 + \frac{n}{2}\right)$$

Hints: (a) recall the series expansion $\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}$; (b) you will likely need to use substitution twice; (c) recall that for some function $h_n(t)$, if $\int_a^b \sum_{n=0}^{\infty} |h_n(t)|dt < \infty$, then

$$\int_a^b \sum_{n=0}^{\infty} |h_n(t)|dt = \sum_{n=0}^{\infty}\left(\int_a^b |h_n(t)|dt\right).$$

(b) Using the moment generating function M_W, find $\mathrm{Var}(W^2)$.