

Homework 5

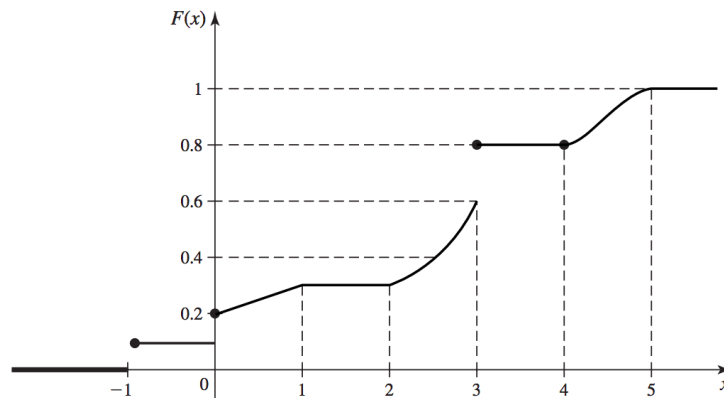
STA 4321/5325, Fall 2019, MWF 8:30am

Professor: Aaron J. Molstad

Due date: Wednesday, October 16th, 2019

All work must be shown for complete credit. W.M.S. denotes the course textbook (*Mathematical Statistics with Applications*). For clarity, we use $\exp(x) \equiv e^x$ interchangeably. **Problem #2 is optional and worth 2pts of extra credit.**

1. Let the following be the probability distribution function for the random variable X .



Compute the following probabilities. (Notice that there are points of discontinuity, so that for this random variable, it is not true that $P(X \leq a) = P(X < a)$ for all a .)

- | | | | |
|--------------------------|--------------------|--------------------------|--------------------------|
| (a) $P(X = -1)$ | (b) $P(X < 0)$ | (c) $P(X \leq 0)$ | (d) $P(X = 1)$ |
| (e) $P(0 \leq X \leq 3)$ | (f) $P(0 < X < 3)$ | (g) $P(1 < X \leq 2)$ | (h) $P(1 \leq X \leq 2)$ |
| (i) $P(X > 5)$ | (j) $P(X \geq 5)$ | (k) $P(3 \leq X \leq 4)$ | |

2. **(Optional)** Let Y be a random variable following the binomial distribution based on n independent Bernoulli trials, each with success probability p .

- (a) Compute $E[Y(Y-1)(Y-2)]$
(b) Compute $E[(Y+1)^{-1}]$

3. Suppose Y is a negative binomial random variable with parameters r and p (i.e., Y is the number of failures before the r th success in a sequence of Bernoulli trials, each with success probability p). Define $\tilde{Y} = Y + r$. Then, \tilde{Y} can be interpreted as the number of the trial on which the r th success is observed.

- (a) What is the probability mass function of \tilde{Y} ?
(b) Using your answer to (a), compute $E(\tilde{Y})$.
(c) Is the variance of \tilde{Y} the same as the variance of Y ? How do you know?

4. Let Z be a random variable with probability density function

$$f_Z(z) = c \cdot \exp\left(-\frac{z^2}{2}\right), \quad z \in (-\infty, \infty).$$

Determine c so that f_Z is a valid probability density function. **Note:** do not try to solve the needed integral by hand – you may use WolframAlpha or some other online resource.

5. Suppose that the random variable X has probability density function

$$f_X(x) = \begin{cases} \frac{4}{3}(1 - x^3) & : 0 < x < 1 \\ 0 & : \text{otherwise} \end{cases}$$

- (a) Compute the probability distribution function F_X .
- (b) Sketch the probability density function f_X .
- (c) Evaluate the following probabilities: $P(X < 1/2)$, $P(1/4 < X < 3/4)$, $P(X > 1/3)$

6. Suppose that the probability distribution function of a random variable X is as follows:

$$F_X(x) = \begin{cases} \exp(x - 3) & : x \leq 3 \\ 1 & : x > 3 \end{cases}$$

Find and sketch the probability density function f_X .

7. Suppose that a continuous random variable X takes values over the interval $[a, b]$ (i.e., $\mathfrak{X} = \text{Range}(X) = [a, b]$). Furthermore, suppose X takes values such that every interval of the same length has equal probability (assuming the interval is entirely contained in $[a, b]$). That is,

$$P(c < X < c + h) = P(d < X < d + h)$$

for all c, d , and $h \in \mathbb{R}$ such that $[c, c + h] \subseteq [a, b]$ and $[d, d + h] \subseteq [a, b]$. What is the probability density function for X ?