1. If k people are seated in a random manner in a *circle* containing n chairs ($n > k$), what is the probability that the people will occupy k adjacent chairs in the circle? [**Hint**: Note that a circle has no beginning or end. So, for $k = 2$, along with the usual $(n-1)$ pairs we have when the chairs are on a row (as discussed in class), in this case we have one more pair, namely, $(n, 1)$. What happens for general k (i.e., $k \geq 2$)? Drawing a circle and plotting n points on it might help you better visualize the problem.]

   Solution. There are $\binom{n}{k}$ possible sets of k seats to be occupied, and they are all equally likely. A circle has no end or beginning, so there are n possible k-tuples of adjacent seats. (The last k-tuple is $(n, 1, 2, \cdots, k-1)$.) Hence, the required probability is

   $$\frac{n}{\binom{n}{k}} = \frac{(n-k)!\,k!}{(n-1)!}.$$

   □

2. (WMS, Problem 2.53.) Five firms, $F_1, F_2, ..., F_5$, each offer bids on three separate contracts, C_1, C_2, and C_3. Any one firm will be awarded at most one contract. The contracts are quite different, so an assignment of C_1 to F_1, say, is to be distinguished from an assignment of C_2 to F_1.

   (a) How many sample points are there altogether in this experiment involving assignment of contracts to the firms? (No need to list them all.)

   (b) Under the assumption of equally likely sample points, find the probability that F_3 is awarded a contract.

   Solution. (a) In choosing three of the five firms, order is important since the contracts are different. So, there are $P_3^5 = 60$ sample points in total.

   (b) Let $A = \{F_3 \text{ is awarded a contract}\}$. First fix a contract out of three for F_3. This can be done in 3 ways. Now, if F_3 is awarded one contract, the remaining two contracts are awarded to the other 4 firms. This can be done in $P_2^4 = 12$ ways. Hence, $\#(A) = 3 \times 12 = 36$. So, $P(A) = 36/60 = 3/5$.

   □

3. (WMS, Problem 2.64.) A balanced die is tossed six times, and the number on the uppermost face is recorded each time. What is the probability that the numbers recorded are 1, 2, 3, 4, 5, and 6 in any order (i.e., each of the six possible numbers appears)?

   Solution. Because each die can result in 6 different outcomes, the total number of sample points is 6^6. Let $A = \{\text{the numbers recorded are 1, 2, 3, 4, 5, and 6 in any order}\}$. Then A contains all possible permutations of the six numbers $1, \ldots, 6$. Therefore $\#(A) = 6!$ and $P(A) = 6!/6^6 = 5/324$. □

4. An exam consists of 10 multiple choice questions, each with two possible answers - one correct and one incorrect. A student is trying to answer the questions by guessing at random.

(a) What is the probability that the student gets at most two correct answers?

(b) What is the probability that she gets at least eight correct answers?

Solution. (This problem is very similar to the coin flipping example discussed in class, where we were interested in the probability of getting 3 heads in a sequence of 10 flips of a balanced coin.) Since each question has two possible answers, total number of ways the 10 questions can be answered is $2^{10} = 1024$. Each outcome is equally likely, since the student is guessing the answers at random, and the number of ways the student gets exactly i correct answers is $\binom{10}{i}$, for $i = 0, \ldots, 10$.

(a) Let $A = \{$The student gets at most two correct answers$\}$. Then

$$\begin{aligned} P(A) &= \text{P(The student gets 0 correct answer or 1 correct answer or 2 correct answers)} \\ &= \frac{\binom{10}{0} + \binom{10}{1} + \binom{10}{2}}{1024} = \frac{56}{1024}. \end{aligned}$$

(b) Let $B = \{$The student gets at least eight correct answers$\}$. Then

$$P(B) = \frac{\binom{10}{8} + \binom{10}{9} + \binom{10}{10}}{1024} = \frac{56}{1024}.$$

□

5. Suppose that 100 mathematics students are divided into five classes, each containing 20 students, and that awards are to be given to 10 of these students. If each student is equally likely to receive an award, what is the probability that exactly two students in each class will receive awards?

Solution. The order among students receiving the awards are not important, since there is no distinction among the awards. There are $\binom{100}{10}$ ways of choosing 10 from 100 mathematics students. For each of the 5 classes, there are $\binom{20}{2}$ ways of choosing two from 20 students. Therefore, total number of ways of choosing two students from each class is $\binom{20}{2}^5$. Since all outcomes are equally likely, required probability is $\binom{20}{2}^5 / \binom{100}{10} = (190)^5 / \binom{100}{10} = (190)^5 \, 10! \, 90!/100!$. □

6. Each of 50 families has two children. A group of 50 children is chosen at random.

(a) What is the probability that a given family is represented?

(b) What is the probability that all families are represented?

Solution. There are $\binom{100}{50}$ ways of choosing 50 children from $50 \times 2 = 100$ children. All outcomes are equally likely, since the children are chosen at random.

(a) Let $A = \{$The given family is represented$\}$. Then $\bar{A} = \{$The given family is NOT represented$\}$. Hence, $\bar{A}$ consists of all sample points where the 50 children are chosen from the rest of the 49 families, i.e., from $49 \times 2 = 98$ children. Hence $\#(\bar{A}) = \binom{98}{50}$, which means $P(\bar{A}) = \binom{98}{50} / \binom{100}{50}$, since all outcomes are equally likely. Therefore,

$$P(A) = 1 - P(\bar{A}) = 1 - \frac{\binom{98}{50}}{\binom{100}{50}} = 1 - \frac{98!}{50! \, 48!} \cdot \frac{50! \, 50!}{100!} = 1 - \frac{49 \times 50}{99 \times 100} = \frac{149}{198}.$$

(b) This is similar to problem 5. If all families are represented, then one child from each familiy must be chosen. Because there are two children in each family, a single child from one family can be chosen in $\binom{2}{1}$ ways. Hence, a single child from each of the 50 families can be chosen in $\binom{2}{1}^{50} = 2^{50}$ ways. Therefore, required probability is $2^{50}/\binom{100}{50} = \frac{2^{50}\,50!\,50!}{100!}$.

□

7. A deck of 52 cards contains four aces. If the cards are shuffled and distributed in a random manner to four players so that each player receives 13 cards, what is the probability that all four aces will be received by the same player?

Solution. There are (at least) two ways to solve this problem.

Method 1. Call the four players A, B, C and D. Here we view the problem (or the corresponding experiment) as that of distributing (partitioning) 52 cards into four groups (group 1 for A, group 2 for B and so on), each with 13 cards. There are

$$\binom{52}{13\ 13\ 13\ 13} = \frac{52!}{13! \times 13! \times 13! \times 13!}$$

ways of doing this. Now if A receives all 4 aces, then the remaining $(52 - 4) = 48$ cards are to be distributed among A, B, C, D, so that A gets $(13 - 4) = 9$ cards and each of B, C and D gets 13 cards. There are $\binom{48}{9\ 13\ 13\ 13}$ ways of doing this. Similarly, B can receive all 4 aces in $\binom{48}{13\ 9\ 13\ 13}$ ways, and so on. Hence, total number of ways of distributing 52 cards into four players, each receiving 13 cards, in such a manner that all four aces are received by the same player is

$$\binom{48}{9\ 13\ 13\ 13} + \binom{48}{13\ 9\ 13\ 13} + \binom{48}{13\ 13\ 9\ 13} + \binom{48}{13\ 13\ 13\ 9} = 4\,\frac{48!}{9! \times 13! \times 13! \times 13!}.$$

Therefore, the required probability is

$$4 \cdot \frac{48!}{9! \times 13! \times 13! \times 13!} \cdot \frac{13! \times 13! \times 13! \times 13!}{52!} = \frac{4.10.11.12.13}{49.50.51.52} = \frac{10.11.12}{49.50.51} \approx 0.01.$$

Method 2. Note that we can also view the given problem as that of choosing positions for the four aces in the deck (since the random experiment is not described explicitly, both interpretations are valid). Each card occupies a single position, and out of the 52 available positions we need to choose 4 for placing the 4 aces. This can be done in $\binom{52}{4}$ ways. Now choose one player out of four, in $\binom{4}{1} = 4$ ways. In order for that (chosen) player to receive all four aces, the aces must be placed among the 13 cards received by that player, which can be done in $\binom{13}{4}$ ways. Hence, total number ways of choosing the positions so that all four aces are received by the same player is $4\binom{13}{4}$. Therefore, the required probability is

$$4\frac{\binom{13}{4}}{\binom{52}{4}} = \frac{4.13.12.11.10}{52.51.50.49} = \frac{10.11.12}{49.50.51} \approx 0.01.$$

□

8. (WMS, Problem 2.68.) Show that, for any integer $n \geq 1$,

   (a) $\binom{n}{0} = 1$. Interpret this result.
   (b) $\binom{n}{n} = 1$. Interpret this result.
   (c) $\binom{n}{r} = \binom{n}{n-r}$, for any $r = 0, 1, \cdots, n$. Interpret this result.
   (d) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$. [**Hint**: Consider the binomial expansion of $(x+y)^n$ with $x = y = 1$.]

Solution. (a) $\binom{n}{0} = \frac{n!}{0!\,n!} = 1$. (Since $0! = 1$.) There is only one way to choose all of the items.

(b) $\binom{n}{n} = \frac{n!}{n!\,0!} = 1$. There is only one way to chose none of the items.

(c) $\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \binom{n}{n-r}$. The number of ways of choosing r out of n objects is the same as that of choosing $n - r$ out of n objects.

(d) $2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = \sum_{i=0}^{n} \binom{n}{i}$.

□

9. (WMS, Problem 2.69.) Prove that, for all positive integers n and k ($n \geq k$), $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Solution. Note that

$$\begin{aligned}\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!} \\ &= \frac{n!\,(n-k+1)}{k!\,(n-k+1)!} + \frac{n!\,k}{k!\,(n-k+1)!} \\ &= \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}.\end{aligned}$$

□

10. A company has 20 new jobs for which it has recruited 20 employees. There are 6 jobs in City 1, 4 jobs in City 2, 5 jobs in City 3 and 5 jobs in City 4. Out of the 20 employees, 4 are friends. Assuming that the company gives no preference to any person in assigning jobs, find the probability that all 4 friends land in the same city.

Solution. Note that the total number of ways the 20 jobs can partitioned be into four groups (group i for City i, $i = 1, 2, 3, 4$) of 6, 4, 5 and 5 jobs is $\binom{20}{6,\,4,\,5,\,5}$. This is the total number of sample points. The total number of sample points with all four friends landing in City 1, City 2, City 3 and City 4 are $\binom{16}{2,\,4,\,5,\,5}$, $\binom{16}{6,\,0,\,5,\,5}$, $\binom{16}{6,\,4,\,1,\,5}$ and $\binom{16}{6,\,4,\,5,\,1}$ respectively. Hence,

$$\begin{aligned}P(\text{All 4 friends land in the same city}) &= \frac{\binom{16}{2,\,4,\,5,\,5} + \binom{16}{6,\,0,\,5,\,5} + \binom{16}{6,\,4,\,1,\,5} + \binom{16}{6,\,4,\,5,\,1}}{\binom{20}{6,\,4,\,5,\,5}} \\ &= \frac{\frac{16!}{2!\,4!\,5!\,5!} + \frac{16!}{6!\,5!\,5!} + \frac{16!}{6!\,4!\,5!} + \frac{16!}{6!\,4!\,5!}}{\frac{20!}{6!\,4!\,5!\,5!}}.\end{aligned}$$

□